

Extra Dimensions and Inflation

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It has been proposed that, without invoking supersymmetry, it is possible to solve the hierarchy problem provided the fundamental scale in the higher dimensional theory is at a much lower scale than the Planck scale. In this paper we consider a toy model where we allow $4 + d$ dimensions to evolve from a region determined by the new fundamental scale, which is in our case is electro-weak scale. We further investigate whether it is possible to inflate not only the 3 uncompactified dimensions but also the extra dimensions. We require around 70 e-foldings to stabilize the size of two extra dimensions from M_{EW}^{-1} to the required 1 mm size. This can be easily achieved during inflation, but once inflation ends their evolution is governed by the dynamical history of the universe in a similar way that of Brans-Dicke field in generalized Einstein theories. We also show that we achieve the right level of density contrast without invoking any specific potential. The density contrast depends on the number of extra dimensions and upon the amplitude of the potential.

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I. INTRODUCTION

The longstanding hierarchy problem, that Higgs fields have mass $m_H \sim 1$ TeV and not the Planck mass M_P although through the loop corrections one expects the correction should be of the order of the cut off which is $M_P \sim 10^{18}$ GeV, is solved by supersymmetry by cancelling the loop contribution from fermions and bosons to render such low mass for the Higgs fields. However, this problem can be solved without invoking supersymmetry, as shown in Ref. [1]. If one begins with the higher dimensional theory as a fundamental theory and also considers the scale of gravity to be gauge unification scale instead of M_P in $4 + d$ dimensions, where d represents the number of extra dimensions, then it is possible to recover M_P in the $3 + 1$ dimensional wall where the standard model fields are distributed. Such a model seriously undermines the structure formation if the evolution of extra dimensions are not taken into account [2]. The importance of extra dimensions in dynamical evolution has been discussed in various papers [3,4]. Here we follow a similar approach, allowing $4 + d$ dimensions to evolve from the electro-weak scale. We consider the matter lagrangian in $4 + d$ dimensions and then we reduce the lagrangian to 4 dimensions by taking a simple ansatz for the extra dimensions to be compactified on d dimensional torus. At no stage of our calculation do we consider any specific potential for the matter lagrangian. All we assume is that the potential has a global minimum. By considering this we show that it is possible to inflate the 3 spatial uncompactified dimensions exponentially in the string frame and the extra compactified dimensions to grow from the M_{EW}^{-1} scale to 1mm size required by the two extra dimensions. To evolve the extra dimensions we require a minimum 70 e-foldings of inflation. Once inflation ends the evolution of the extra dimensions is governed by the dynamical history of the universe simi-

lar to the Brans-Dicke field. We also consider the density perturbation created during the inflationary phase in the Einstein frame and conclude that it is easy to produce the required density contrast by constraining the amplitude of the potential from the COBE data. It is worth mentioning that our density perturbation results depend on the number of extra dimensions and also upon the amplitude of the potential.

II. EQUATIONS OF MOTION

We consider our lagrangian in $4 + d$ dimensions:

$$S = \int d^{4+d}X \sqrt{-G} \left[\frac{1}{2\hat{\kappa}_{4+d}^2} \hat{R} + L_{\text{inf}} \right], \quad (1)$$

where G and $\hat{\kappa}_{4+d}^2$ are the $4 + d$ dimensional metric and gravitational constant. L_{inf} is the lagrangian for the scalar field with potential in $4 + d$ dimensions. By taking an ansatz that the line-element in $4 + d$ dimensions takes the following form:

$$d\hat{s}^2 = dt^2 - a(t)^2 g_{\mu\nu} dx^\mu dx^\nu - b(t)^2 g_{ij} dx^i dx^j, \quad (2)$$

where a is the scale factor of the 3 dimensions and b is the scale factor of the extra dimensions. The geometry of g_{ij} is assumed to be torus with a unit volume. It is possible to reduce Eq.(1) to a 4 dimensional lagrangian. In 4 dimensions the lagrangian mimics the lagrangian of Jordan-Brans-Dicke theory where there is an extra field related to the size of the extra dimensions coupled to the Ricci scalar in 4 dimensions. For details, see [5] and [6]. The reduced lagrangian in 4 dimensions in the string frame is:

$$S = \int d^4x \sqrt{-g} \left[-\Phi R - \frac{d-1}{d} \frac{(\nabla\Phi)^2}{\Phi} + 2\kappa^2 \Phi \frac{1}{2} (\nabla\chi)^2 - 2\kappa^2 \Phi V(\chi) \right], \quad (3)$$

where κ and g are the 4 dimensional gravitational constant and the metric. κ is related to $4+d$ dimensional gravitational constant by

$$\kappa^2 = \frac{\hat{\kappa}_{4+d}^2}{2^d b_0^d \pi}. \quad (4)$$

χ and $V(\chi)$ are the corresponding inflaton and its potential in the reduced 4 dimensions and Φ is defined by the scale of compactification.

$$\Phi \equiv \frac{1}{2\kappa^2} \left[\frac{b}{b_0} \right]^d, \quad (5)$$

where b_0 is the scale factor of the extra dimensions at present. we must mention that the $4+d$ dimensional gravitational constant can be recast according to the demand of the newly proposed scale:

$$\hat{\kappa}_{4+d}^2 = \frac{8\pi}{M_{EW}^{2+d}}. \quad (6)$$

In the denominator of Eq.(6), instead of M_p we have a new fundamental scale determined by the electro-weak unification scale M_{EW} . Hence, the 4 dimensional gravitational constant can be expressed by:

$$\kappa^2 = \frac{8\pi}{2^d b_0^d \pi M_{EW}^{(2+d)}}. \quad (7)$$

For our purpose we take b_0 to be 1mm for 2 extra dimensions if $M_{EW} \sim 1\text{TeV}$ as proposed in Ref. [1]. It is worth mentioning at this point that the Planck mass in 4 dimensions is determined by:

$$M_p^2 = 2^d b_0^d \pi M_{EW}^{(2+d)}. \quad (8)$$

We may now write down the equations of motion corresponding to Eq.(3) in the string frame and later on we shall analyze in the Einstein frame also. In a spatially flat Robertson-Walker universe:

$$H^2 + H \frac{\dot{\Phi}}{\Phi} = -\frac{1}{6} \left[\frac{\dot{\Phi}}{\Phi} \right]^2 + \frac{\kappa^2}{3} \left[\frac{1}{2} \dot{\chi}^2 + V(\chi) \right], \quad (9)$$

$$\ddot{\Phi} + 3H\dot{\Phi} = \frac{2\kappa^2\Phi}{1+2/d} [V(\chi) - \dot{\chi}^2], \quad (10)$$

$$\ddot{\chi} + 3H\dot{\chi} = -V'(\chi). \quad (11)$$

An overdot denotes derivative with respect to time and prime with respect to the inflaton field χ . H is the Hubble constant of the universe, $H = \dot{a}/a$. Irrespective of the form of the potential $V(\chi)$, if inflation occurs then

following the slow-roll approximation * one can reduce the above Eqs. (9-11).

$$H^2 \approx \frac{\kappa^2}{3} V(\chi), \quad (12)$$

$$3H \frac{\dot{\Phi}}{\Phi} \approx \frac{2\kappa^2 V(\chi)}{1+2/d}, \quad (13)$$

$$3H\dot{\chi} \approx -V'(\chi). \quad (14)$$

Solving Eqs.(12-13) we get the following solutions [6]:

$$\frac{\dot{\Phi}}{\Phi} = d \frac{\dot{b}}{b} = \frac{2d}{2+d} \frac{\dot{a}}{a}, \quad (15)$$

$$a \propto e^{Ht}. \quad (16)$$

Where we have used Eq.(5) in Eq.(15) to derive the most important relation between the scale factors which enable us to stabilize the extra dimensions. Eq.(16) clearly shows that the scale factor in the uncompactified 3 dimensions grow exponentially fast provided the potential $V(\chi)$ is sufficiently flat and treated almost as a constant during the inflationary phase. This also suggests that the extra dimensions are also inflating but their rate of expansion solely depends on the number of extra dimensions governed by Eq.(15).

Now let us analyze the Hubble parameter during inflation. Before that we must note that the effective potential in 4 dimensions can at most take the value $b_0^d M_{EW}^{4+d}$ and not just M_{EW}^4 †. Here we must mention that had we introduced the inflaton in the 4 dimensional lagrangian we would have got the latter bound on $V(\chi)$. Here we get the following relation:

$$H^2 \approx \frac{8}{3 \cdot 2^d b_0^d M_{EW}^{(2+d)}} M_{EW}^{(4+d)} b_0^d \sim \frac{8M_{EW}^2}{3 \cdot 2^d}. \quad (17)$$

This is an important result as it suggests that the Hubble parameter can roughly take the electro-weak scale in the string frame. Now it is worth questioning the fate of the extra dimensions. In fact our simple analysis shows that the extra dimensions will also grow exponentially fast but they will never be able to overcome the 1 mm size as expected for two extra dimensions, because there

* Appropriate slow-rollover approximations are $|\frac{\dot{\Phi}}{\Phi}| \ll H$, $|\ddot{\Phi}| \ll 3H\dot{\Phi}$, $|\dot{\chi}^2| \ll V(\chi)$, and $|\ddot{\chi}| \ll 3H\dot{\chi}$.

† In general, the potential $V(\chi)$ would be of the order of M_{EW}^4 had we neglected the dynamics of extra dimensions, but a careful analysis suggests that on dimensional grounds $V(\chi) \approx M^4 \equiv b_0^d M_{EW}^{4+d}$; for details, see Eqs. (3.4) and (3.5) in [5]. It is extremely important to note that in 4 dimensions the effective potential should have significant energy contribution from the extra dimensions. This can be vividly seen by arguing that if we do not allow the extra dimensions to evolve, $b \approx b_0$, we get the energy density in 4 dimensions, $V(\chi) \approx M_{EW}^4$.

is a vital difference between the rate of expansions of the extra dimensions and the observable uncompactified universe.

Once inflation ends in this model the χ field oscillates at the bottom of the potential and it can be shown with the help of Eq.(10) that on average the right hand side of the Eq.(10) vanishes, leading to a late time attractor solution for $\Phi = \text{Constant}$, the constant value of Φ is fixed by the present value of the Newton's constant and thus stabilizes the size of the extra dimensions which should be of the order of 1mm. for two extra dimensions. This leads to a novel dynamical method to stabilize the extra dimensions. Hence, in our model the present value of the Newton's constant is fixed right after the end of inflation and during the subsequent evolution of the universe, the value of the Newton's constant remain unchanged.

III. DENSITY PERTURBATION

We carry out the density perturbation calculation in the Einstein frame. In fact the string frame is related to the Einstein frame through the conformal transformation $g_{\mu\nu} = (2\kappa^2\Phi)^{-1}\tilde{g}_{\mu\nu}$ and in terms of field:

$$\kappa\phi = \left[\frac{d+2}{2d}\right]^{1/2} \ln [2\kappa^2\Phi] , \quad (18)$$

where ϕ is the field in the Einstein frame corresponding to the field Φ in the string frame. The reduced 4 dimensional lagrangian in the Einstein frame is:

$$S = \int d^4x \sqrt{-\tilde{g}} \left[-\frac{1}{2\kappa^2} \tilde{R} + \frac{1}{2} (\tilde{\nabla}\phi)^2 + \frac{1}{2} (\tilde{\nabla}\chi)^2 - e^{-\beta\kappa\phi} V(\chi) \right] , \quad (19)$$

where β in our case is defined by the following relation:

$$\beta = \left[\frac{2d}{d+2}\right]^{1/2} . \quad (20)$$

The advantage of the conformal transformation is that it allows the density spectrum and the reheating temperature to be derived using the well known results from the Einstein gravity [7] and [8]. It is easy to derive the slow-roll equations of motion in the Einstein frame by neglecting the kinetic and the double time derivative terms.

$$3\tilde{H}\dot{\phi} \approx \beta\kappa e^{-\beta\kappa\phi} V(\chi) , \quad (21)$$

$$3\tilde{H}\dot{\chi} \approx -e^{-\beta\kappa\phi} V'(\chi) , \quad (22)$$

$$\tilde{H}^2 \approx \frac{\kappa^2}{3} e^{-\beta\kappa\phi} V(\chi) . \quad (23)$$

Here the dot is differentiation with respect to the time in the Einstein frame. It is worth mentioning that the Hubble parameter in the Einstein frame will not be of the order of electro-weak scale but it is modified by the

conformal factor. In fact it is very easy to show with the help of Eq.(5) and Eq.(18) that the Hubble parameter in the Einstein frame is close to the Planck scale in 4 dimensions. This ensures that the number of e-foldings in both the frames are roughly going to be equal[‡]. To carry out the density perturbation calculation we simply assume that one of the scalar fields dominates the other and we also use the above equations Eq.(21-23) to derive the form of the density contrast. In the Einstein frame when $\dot{\phi} \ll \dot{\chi}$:

$$\frac{\delta\rho}{\rho} = \frac{\tilde{H}^2}{2\pi\dot{\chi}} = \frac{\kappa^3}{2\pi\sqrt{3}} \frac{V_h^{3/2}}{V'_h} \left[\frac{b_h}{b_0}\right]^{-d/2} , \quad (24)$$

and, when $\dot{\chi} \ll \dot{\phi}$:

$$\frac{\delta\rho}{\rho} = \frac{\tilde{H}^2}{2\pi\dot{\phi}} = \frac{\kappa^2}{2\pi\sqrt{3}\beta} \sqrt{V_h} \left[\frac{b_h}{b_0}\right]^{-d/2} , \quad (25)$$

where subscript h denotes that the corresponding quantities are evaluated at the time of horizon crossing. We assume that the amplitude of the inflaton potential is determined by a dimensionless parameter α , such that $V(\chi) \propto \alpha b_0^d M_{\text{EW}}^{4+d}$. The detailed form of the potential is not required for our analysis. Further assuming that the compactification scale during horizon crossing is very close to the electro-weak scale, one can easily deduce the density contrast for two different cases. Eq.(24) reduces to a simpler form:[§]

$$\frac{\delta\rho}{\rho} = \frac{(8\alpha)^{3/2}}{2^{3d/2+1}\sqrt{3}\pi} , \quad (26)$$

and Eq.(25) reduces to:

$$\frac{\delta\rho}{\rho} = \frac{8\alpha^{1/2}}{2^{d+1}\sqrt{3}\pi \left[\frac{2d}{d+2}\right]^{1/2}} . \quad (27)$$

The value of the amplitude of the potential is to be adjusted to give the right level of perturbations when our present Hubble scale crossed outside the horizon during inflation. The perturbations observed by COBE satellite

[‡] Number of e-foldings defined in the string frame: $N = \int dt H$, and in the Einstein frame $\tilde{N} = \int d\tilde{t} \tilde{H}$. Now, from the conformal transformation $H = \Omega \tilde{H} - \dot{\Omega}$. In our case $\Omega^2 = 2\kappa^2\Phi$. If $|\frac{\dot{\Omega}}{\Omega H}| \ll 1$, then, $H \approx \Omega \tilde{H}$ and subsequently we find $N \approx \tilde{N}$. In our case $|\frac{\dot{\Omega}}{\Omega H}| = \frac{d}{d+2} \frac{H}{\tilde{H}} \approx \frac{d}{d+2} \frac{M_{\text{EW}}}{M_{\text{p}}} \approx 10^{-17} \ll 1$. This ensures that the observable quantities in both the frames are equivalent and can be translated from one frame to the other.

[§] $V'(\chi) \sim V(\chi)/\chi$, and from dimensional analysis $\chi \approx M \equiv b_0^{d/2} M_{\text{EW}}^{1+d/2}$, we get $V'(\chi) \approx b_0^{d/2} M_{\text{EW}}^{3+d/2}$. For details, see Eqs.(3.5) and (3.7) in [5].

require $\delta\rho/\rho \approx 2 \times 10^{-5}$; this determines the value of α in our case; $\alpha = 1.1 \times 10^{-3}$ when $\dot{\phi} \ll \dot{\chi}$, and $\alpha = 1.2 \times 10^{-10}$ while taking the opposite limit for $d = 2$. For a quadratic potential in 4 dimensions (a quadratic potential enjoys the dimensional consistency in 4 dimensions as well as in higher dimensions) the mass of the inflaton requires $\alpha^{1/2} m_\chi \sim 0.03 M_{\text{EW}}$ when $\dot{\phi} \ll \dot{\chi}$, suggest that the inflaton mass is very close to the electro-weak scale. At this point it is worth mentioning that the density contrast result is quite sensitive to the number of extra dimensions and the scale invariance of the density perturbation is already treated in the context of "soft inflation", see [6].

IV. STABILIZING EXTRA DIMENSIONS

Next we consider the issue of stabilizing the size of the extra dimensions. As we have already shown that the 3 uncompactified spatial dimensions grow exponentially fast and through Eq.(15) it is clear that the scale factor for the extra dimensions also grow exponentially fast. Fortunately the rate of expansions for a and b are different and this is solely responsible to stabilize the size of the extra dimensions soon after the end of inflation. We argue that as soon as inflation stops, the extra dimensions stop growing and stabilize close to the observed value b_0 , similar to the Brans-Dicke field which saturates as soon as inflation ends. Solving Eq.(15) we can estimate the evolution of b during inflation.

$$\frac{b_f}{b_i} = \left[\frac{a_f}{a_i} \right]^{\frac{2}{2+d}}, \quad (28)$$

where the subscripts stands for final and initial values. Now we observe that for the extra dimensions to grow from 1 TeV scale which corresponds to 10^{-16}cm to 1mm size, we require at least 35 e-foldings of inflation in the extra dimensions. Hence, the left hand side of the Eq.(28) will be exactly e^{35} . If we assume that there are only two extra dimensions then the right hand side of the Eq.(28) suggests that the scale factor in the uncompactified dimensions will grow e^{70} , which is a reasonable number of e-foldings to solve the usual cosmological problems in the uncompactified dimensions. Here we must say that the analysis is independent of the details of the model and hence we do not require any fine tuning in the parameters of the inflaton potential. It is important to note that for $d > 2$, one requires more than 70 e-foldings of inflation in the uncompactified dimensions to stabilize the extra dimensions. In fact further evolution of the extra dimensions depends on the dynamical evolution of the universe, similarly to the Brans-Dicke case, this claim is further validated by inspecting Eq.(10), that, when the χ field oscillates at the bottom of the potential at the end of inflation and on average cancelling the potential term with the kinetic term, the dynamical stabilization of the Φ field is achieved automatically. In fact this is

a novel way to stabilize the extra dimensions where we need not require to invoke any potential to fix the value of the Newton's constant.

V. CONCLUSIONS

In this paper we have discussed that inflation at the TeV unification scale can occur in a usual sense with the Hubble constant $H^{-1} \approx M_{\text{EW}}^{-1}$ in the string frame. The only fundamental energy density is M_{EW}^{4+d} . the energy density in 4 dimensions is largely contributed by the energy density from the extra dimensions. Our results are quite generic and we hardly require any fine tuning in our potential. Density perturbations in this model depends on the number of extra dimensions and on the amplitude of the potential. We must say that our density perturbation calculation is valid only in the Einstein frame but we have also shown that it is adequate to discuss the perturbation analysis in the Einstein frame because string frame is conformally related to the Einstein frame. We show that indeed it is possible to stabilize the size of the extra dimensions. For two extra dimensions to stabilize at 1mm size one needs to invoke roughly 70 e-foldings of expansion and once inflation ends the fate of the extra dimensions is determined by the dynamical history of the universe, similar to the case of the Brans-Dicke field. We must mention that here we need not invoke any apriori potential to fix the dilaton, rather the dilaton is fixed by the dynamical stabilization to give rise to the value of the Newton's constant depending on the present size of the extra dimensions.

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